Data-driven distributionally robust MPC using the Wasserstein metric

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***Abstract*— Distributionally robust optimization is a technique for decision making under uncertainty where the probability distribution of the uncertain problem is itself subject to uncertainty. A novel data-driven MPC scheme is proposed to control constrained stochastic linear systems using distributionally robust optimization. Distributionally robust constraints based on Wasserstein ball are imposed to bound the expected state constraint violation in the presence of process disturbance. A feedback control law is solved to guarantee that the predicted states comply with constraints with regard to the worst-case distribution within the Wasserstein ball centered at the discrete empirical probability distribution. The resulting distributionally robust MPC framework is tractable and efficient. The effectiveness of the proposed scheme is demonstrated through two numerical case studies.**

I. INTRODUCTION

Model predictive control (MPC) has demonstrated remarkable success due to its ability to handle multivariate dynamics and constraints[1], [2]. MPC solves an open-loop optimal control problem at each sampling time based on a nominal model to decide a finite sequence of control actions from which the first element of the sequence is implemented [3].

In the context of control under uncertainty, two important methodologies arise to guarantee constraint satisfaction: Robust MPC (RMPC) [4] and stochastic MPC (SMPC). The former addresses the receding horizon optimal control problem for uncertain systems in a deterministic fashion by assuming bounded uncertainties and providing solutions for the worst case-scenario. Some important approaches to RMPC are min-max optimization [5] and tube-based MPC [6]. However, some worst-case scenarios are unlikely to work in practice; as resulting control designs tend to be over conservative or even infeasible [7]. In addition to conservativeness, if the assumed uncertainty sets are inaccurate, the controller may have poor performance. To address this, approaches have been proposed to reduce conservativeness in the context of RMPC, e.g. [8], [9], [10].

In contract to RMPC, SMPC solves a stochastic optimization problem by assuming distributional information of the uncertainty [11]. Chance constraints on SMPC reduce the inherent conservativeness of robust MPC via the trade-off between constraint satisfaction and closed-loop performance [12]. However, deviation of the assumed distribution from the true one caused by poor assumptions or limited available data may result in sub-optimality, infeasibility and unwanted behavior of the system [13].

To overcome the conservativeness of RMPC and the distributional mismatch of SMPC, we explore a distributionally robust optimization approach - distributionally robust MPC. In distributionally robust optimization (DRO), a variant of the stochastic optimization is explored where the probability distribution is itself uncertain. DRO minimizes an expected loss function, where the expectation comes from the worst-case probability distribution of an ambiguity set.

Just as wrong assumptions on the distribution of the uncertainty can be detrimental to the objective's performance on an MPC scheme, chance constraints can also be affected by this mismatch, incurring in severe violations. As a counterpart to chance constraints, distributionally robust chance constraints assume the actual distribution of uncertain variables belongs to an ambiguity set. This ambiguity set contains all distributions with a predefined characteristic (e.g. first or second moments), and such an ambiguity set can be computed from historical data. Distributionally robust constraints have direct connection to the constraints incorporated in the classical paradigms of RMPC and SMPC [14]. To capture a decision maker's attitude towards risk and ambiguity, distributionally robust constraints [15] are considered in this work. Furthermore, we characterize the ambiguity set as a Wasserstein ball around an empirical distribution with a radius defined by the Wasserstein metric [16].

The Wasserstein ambiguity set has received increasing attention in distributionally robust optimization due to its modeling flexibility, finite-sample guarantee and tractable reformulation into convex programs [17]. In contrast to a Wasserstein ambiguity set, other ambiguity sets do not enjoy these important properties. Specifically, approaches leveraging moment-based ambiguity do not have finite-sample guarantees [18] and ambiguity sets using ϕ-divergence such as Kullback–Leibler divergence typically contain irrelevant distributions [19]. These drawbacks motivate our use of a Wasserstein ambiguity set.

Distributionally robust optimization in control problems has been studied with regard to different formulations of objective functions. In the setting of multi-stage stochastic optimal power flow problem, a framework is proposed to solve multi-stage feedback control policies with respect a

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Conditional Value at Risk (CVaR) objective [20]. A control policy for wind power ramp management is solved via dynamic programming by reformulating the distributionally robust value function with a tractable form: a convex piecewise linear ramp penalty function [21]. For linear quadratic problems, a deterministic stationary policy is determined by solving an data-irrelevant discrete algebraic Riccati equation [22]. The control policies for both finite horizon optimal control problem with expected quadratic objective and infinite horizon optimal control problem with an average cost can be determined concerning distributionally robust CVaR constraints modeled by moment-based ambiguity sets [23]. Recently, a data-driven distributionally robust MPC with a moment-based ambiguity set for quadratic objective function under multi-stage risk measures was proposed in [24].

A. Main Contribution

Compared to existing related studies which consider linear objective function or moment-based ambiguity sets, our work uniquely considers a finite-horizon control problem with distributionally robust constraints constructed by Wasserstein ambiguity set. The main contributions of this paper are summarized as follows:

1. A distributionally robust optimal control problem with distributionally robust chance constraints and expected quadratic objective is proposed to determine purified-output-based (POB) affine control laws.

2. A practical Algorithm is proposed, which results in a tractable conic optimization problem.

3. Finite sample guarantee of chance constraints is proved and demonstrated via a case study.

B. Notation

Let $x_{[k,k+N]}$ denote the concatenated state vector $[x_k^\top, x_{k+1}^\top, \dots, x_N^\top]^\top$ and $[x_{[k,k+N]}]_i$ be the i-th entry of the vector. $\{[x_{[k,k+N]}]_i\}_{i=a}^b$ denote a sequence of a-to-b-th entries from the concatenated state vector. Let $\mathrm{Tr}(\cdot)$ be the trace operator. We denote by $\mathbb{S}_+^n$ and $\mathbb{S}_{++}^n$ the sets of all positive semidefinite and positive definite symmetric matrices in $\mathbb{R}^{n\times n}$, respectively. The diagonal concatenation of two matrices A and B is denoted by $\mathrm{diag}(A,B)$. $A_{i,j}$ is the entry of i-th row and j-th column in a 2D matrix and $[A]_{i,j:k}$ is the row vector of i-th row and j-th to k-th columns in matrix A. e_j denote a column vector with all entries 0 expect j-th entry equal to 1.

All random vectors are defined as measurable functions on an abstract probability space $(\Omega, X, \mathbb{P})$, where Ω is referred to as the sample space, X represents the σ-algebra of events, and $\mathbb{P}$ denotes the true but unknown probability measure. We denote by δ_ξ the Dirac distribution concentrating unit mass at ξ and by δ_k the state difference of nominal and disturbed system at sampling time k. The N-fold product of a distribution $\mathbb{P}$ on Ξ is denoted by $\mathbb{P}^N$, which represents a distribution on the Cartesian product space Ξ^N. $\mathscr{M}(\Xi)$ is the space of all probability distributions $\mathbb{Q}$ supported on Ξ with finite expected norm. The training data set comprising N_s samples is denoted by $\widehat{\Xi}_{N_s} := \left\{\widehat{\xi}_i\right\}_{i<N_s} \subseteq \Xi$.

C. Organization

The remainder of this paper is organized as follows. Problem formulation of determining POB affine control laws for disturbed systems is introduced in Section II. Preliminaries on distributionally robust control embracing Wasserstein metric and the corresponding optimization problem are covered in Section III. The main results including tractable formulation of distributionally robust optimization problem and finite sample guarantee are discussed in Section IV along with a practical Algorithm. Simulation experiments for case studies mass spring system and inverted pendulum are illustrated in Section V, and the results are also analyzed. Conclusions are summarized in Section VI.

II. PROBLEM FORMULATION

In this section we explain how to derive POB affine control laws [25] for a discrete-time linear time-invariant (LTI) dynamical system with additive disturbance. We consider a discrete-time LTI system at time k

$$\begin{aligned} x_0 &= x \\ x_{k+1} &= Ax_k + Bu_k + Cw_k \\ y_k &= Dx_k + Ew_k, \end{aligned} \tag{1}$$

where state $x_k \in \mathbb{R}^{n_x}$, input $u_k \in \mathbb{R}^{n_u}$, output $y_k \in \mathbb{R}^{n_y}$ and the disturbance $w_k \in \mathbb{R}^{n_w}$. Both process noise and measurement noise are modelled via matrices C and E. Our design target is to enable the closed-loop system of (1) to meet prescribed requirements. One of the requirements is to satisfy polyhedral state constraints $C_p x_{[k,k+N]} \leq D_p$ within the prediction horizon N.

The POB affine control laws are derived based on the discrepancy between the disturbed system and its corresponding nominal system (see Definition 2.2). Without loss of generality, we assume the equilibrium point is at the origin.

Definition 2.1 (Model): Given a disturbed system in the form of (1), we define the corresponding nominal system initialized at the equilibrium point and not disturbed by exogenous inputs as *model*

$$\begin{aligned} \widehat{x}_0 &= 0 \\ \widehat{x}_{k+1} &= A\widehat{x}_k + Bu_k \\ \hat{y}_k &= D\widehat{x}_k. \end{aligned} \tag{2}$$

The open-loop state difference of model and disturbed system $\delta_k = x_k - \widehat{x}_k$ evolves according to the system matrices and disturbances. The accumulated influence of disturbances can be measured via purified outputs $v_t = D\delta_t + Ew_t = y_t - \widehat{y}_t$ where $t \in [k, k+N-1]$. This allows us to consider POB affine control laws based on the history of disturbance and inputs [26].

Hence, we will solve the following problem: Given system matrices, initial state $x_0 = x$ and collected disturbance data points, we determine control laws for prediction horizon N by leveraging a tractable data-driven optimization problem, such that the system can be steered to a desired equilibrium state while satisfying prescribed chance constraints $\mathbb{E}\left[C_p x_{[k,k+N]} \leq D_p\right] \geq 1-\beta$. The parameterized affine control laws will be defined as following.

Definition 2.2 (POB Affine Control Laws): At sampling time t, given purified outputs from k to t, we define *POB affine control laws* as

$$u_t = h_t + \sum_{\tau=k}^{t} H_{t,\tau} v_\tau \tag{3}$$

with $t \in [k, k+N-1]$.

Note that the definition of POB affine control laws above is equivalent to the affine control laws only dependent on initial state and disturbance sequence or only dependent on outputs of disturbed system.

Lemma 1 (Equivalent Control Laws): For every POB affine control laws in the form of (3), there exists control laws resulting in exactly the same closed-loop state-control trajectories dependent only on:

(i) initial state and disturbance sequence

(ii) outputs of disturbed system.

Proof: (i) follows directly from the definition of purified output. We reformulate (3) by inserting the accumulated state discrepancy and the current disturbance. Then we acquire

$$v_t = \sum_{\tau=0}^{t} DA^{t-\tau} C^{\tau} x_0 + E w_t \tag{4}$$

Now we can show that (i) holds after inserting (4) into (3).

The proof of (ii) follows a similar path as the one for theorem in [25, Theorem 14.4.1.]. ■

From lemma 1, it is thus clear that for a disturbed linear system, it is possible to exert control actions computed by affine laws in disturbance to guarantee equivalent closed-loop state trajectories resolved by output feedback controller.

For the convenience of constructing the optimization problem to determine control laws, we derive a compact form of dynamical system applying the POB control laws. The dynamics of the linear system over the finite horizon N can then be written as $x_{k,k+N} = A_x x + B_x u_{[k,k+N-1]} + C_x w_{[k,k+N-1]}$ and the corresponding measurements is formulated as $y_{[k,k+N-1]} = A_y x + B_y u_{[k,k+N-1]} + (C_y + E_y) w_{[k,k+N-1]}$, where $A_y = \begin{bmatrix} DA^0 \\ \vdots \\ DA^{N-1} \end{bmatrix}$, $B_y = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ DA^0B & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 \\ DA^{N-2}B & DA^{N-3}B & \dots & DA^0B & 0 \end{bmatrix}$ and $E_y = \begin{bmatrix} E & & \\ & \ddots & \\ & & E \end{bmatrix}$. Consider now the inputs characterized by POB affine control laws, we derive $u_{[k,k+N-1]} = H_N(\tilde{C}_y + \tilde{E}_y)\tilde{w}_{[k,k+N-1]} = \tilde{H}_N \tilde{w}_{[k,k+N-1]}$, where $\tilde{w}_{[k,k+N-1]} = \begin{bmatrix} 1 & w_{[k,k+N-1]}^\top \end{bmatrix}^\top$ as extended disturbance vector, $\tilde{C}_y = \begin{bmatrix} 0 & 0 \\ A_y x_0 & C_y \end{bmatrix}$ and $\tilde{C}_E = \begin{bmatrix} 1 & 0 \\ 0 & E_y \end{bmatrix}$. Furthermore, we use H_N with subscript N to denote the control laws for prediction horizon N. Finally, this allows us to write the stacked state vector as linear matrix equation

$$\tilde{x}_{[k,k+N]} = \left(B_x \tilde{H}_N + \tilde{C}_x\right) \tilde{w}_{[k,k+N-1]}, \tag{5}$$

where $\tilde{C}_x = \begin{bmatrix} A_x x_0 & C_x \end{bmatrix}$. Our goal is then to determine the control laws $\tilde{H}_N$ which steer the system to the origin whilst guaranteeing distributionally robust constraint satisfaction.

III. DISTRIBUTIONALLY ROBUST MPC

A. Ambiguity Sets and Wasserstein Balls

Distributionally robust optimization is an optimization model where limited information about the underlying probability distribution of the random parameters in a stochastic model is assumed. Therefore, to model distributional uncertainty, we characterize the partial information about the true distribution $\mathbb{P}$ by a set of probability measures on (Ω, X). This set is termed as *ambiguity set* [15]. In this paper, we focus on an ambiguity set specified by a discrepancy model [14] wherein the distance function on the probability distribution space is characterized by the Wasserstein metric. The Wasserstein metric defines the distance between all probability distributions $\mathbb{Q}$ supported on Ξ with finite p-moment $\int_{\Xi} \|\xi\|^p \mathbb{Q}(d\xi) < \infty$.

Definition 3.1 (Wasserstein Metric [27]): The *Wasserstein metric* of order $p \geq 1$ is defined as $d_w : \mathscr{M}(\Xi) \times \mathscr{M}(\Xi) \rightarrow \mathbb{R}$ for all distribution $\mathbb{Q}_1, \mathbb{Q}_2 \in \mathscr{M}(\Xi)$ and arbitrary norm on $\mathbb{R}^{n_\xi}$:

$$d_{\mathrm{W}}\left(\mathbb{Q}_1, \mathbb{Q}_2\right) := \inf_{\Pi} \left\{ \left(\int_{\Xi^2} \|\xi_1 - \xi_2\|^p \, \Pi\left(\mathrm{d}\xi_1,\ \mathrm{d}\xi_2\right) \right)^{1/p} \right. \tag{6}$$

where Π is a joint distribution of ξ_1 and ξ_2 with marginals $\mathbb{Q}_1$ and $\mathbb{Q}_2$ respectively.

Specifically, we define an ambiguity set centered at the empirical distribution leveraging the Wasserstein metric

$$\mathbb{B}_{\varepsilon}\left(\widehat{\mathbb{P}}_{N_s}\right) := \left\{ \mathbb{Q} \in \mathscr{M}(\Xi) : d_{\mathrm{W}}\left(\widehat{\mathbb{P}}_{N_s}, \mathbb{Q}\right) \leq \varepsilon \right\} \tag{7}$$

which specifies the Wasserstein ball with radius $\varepsilon > 0$ around the discrete empirical probability distribution $\widehat{\mathbb{P}}_{N_s}$. The empirical probability distribution $\widehat{\mathrm{P}}_{N_s} := \frac{1}{N_s} \sum_{i=1}^{N_s} \delta_{\widehat{\xi}_i}$ is the uniform distribution on the training data set $\widehat{\Xi}_{N_s} := \left\{\widehat{\xi}_i\right\}_{i \leq N_s} \subseteq \Xi$. $\delta_{\widehat{\xi}_i}$ is the Dirac distribution concentrating unit mass at $\widehat{\xi}_i \in \mathbb{R}^{n_\xi}$. The radius ε should be carefully selected so that the ball can contain the true distribution $\mathbb{P}$ with high fidelity and not unnecessarily large to hedge against over-conservative solutions. The impact of the ball radius will be illustrated in the Section V.

B. Data-Based Distributionally Robust MPC

We now consider the optimal control problem for the system (5) enforcing distributionally robust constraints to be satisfied, i.e.

$$\sup_{\mathbb{Q}\in\mathbb{B}_\varepsilon\left(\widehat{\mathbb{P}}_{N_s}\right)} \mathbb{E}^{\mathbb{Q}}[\ell(\xi,H_N)] \leq U, \tag{8}$$

where ℓ is a function representing state constraints in the aforementioned polyhedral form dependent on affine control laws and disturbance, and U is a prescribed bound. Distributionally robust constraints in a stochastic setting can take the information about the probability distribution into account via (7) such that the prescribed state constraints in an average sense can hold with respect to the worst-case distribution within the ball (7).

Our aim is to find admissible affine control laws with respect to the distributionally robust constraints whilst minimizing an objective J_N. We characterize the objective function as a discounted sum of quadratic stage costs

$$\begin{aligned} J_N(x,H_N) \quad &:= \inf_{\mathbb{Q}\in\mathbb{B}_\varepsilon\left(\widehat{\mathbb{P}}_{N_s}\right)} \mathbb{E}_{\mathbb{Q}}\left\{\sum_{t=k}^{k+N-1}\beta^t\left[x_t^\top Q x_t + u_t^\top R u_t\right]\right. \\ &\left. + \beta^N x_{k+N}^\top Q_f x_{k+N}\right\}, \end{aligned} \tag{9}$$

with $\beta\in(0,1]$ as discount factor. It is further assumed that $Q,Q_f\in\mathbb{S}_+$and $R\in\mathbb{S}_{++}$ so that J_N is convex. We can now formulate the optimal control problem at sampling time k to determine affine control laws within an N-step prediction horizon

$$\begin{aligned} \inf_{H_N} \quad & J_N(x,H_N) \\ \text{s.t.} \quad & x_{t+1} = Ax_t + Bu_t + Cw_t, \quad x_0 = x, \quad \forall t\in[k,k+N] \\ & \sup_{\mathbb{Q}\in\mathbb{B}_\varepsilon\left(\widehat{\mathbb{P}}_{N_s}\right)} \mathbb{E}^{\mathbb{Q}}[\ell_j(\xi,H_N)] \leq U_j, \quad \forall j\leq N_b, \end{aligned} \tag{10}$$

where N_b is the number of constraints imposed on states. This problem with distributionally robust constraints appears to be intractable due to the infinite-dimensional optimization over probability distributions. However, we will demonstrate a tractable reformulation in the next section.

IV. A TRACTABLE CONVEX CONE PROGRAM REFORMULATION

In this section we will rewrite the distributionally robust control problem (10) with type-1 ($p=1$) Wasserstein metric into a finite-dimensional convex cone program leveraging results from robust and convex optimization. After proposing a tractable reformulation, we will introduce a practical data-driven Algorithm to handle the disturbed constrained control problem. The control laws solved as cone programs enjoy the finite sample guarantee, i.e. constraint (8) is satisfied with regard to a specified level of confidence by collecting finite data points.

We first make some assumptions on the random vector ξ and disturbance w_k.

Assumption 1 (i.i.d. Disturbance): We assume that in the discrete-time LTI system (1), the disturbance w_t is an i.i.d. random process with covariance matrix Σ_{w_k} and mean μ_{w_k} for all $t\in\mathbb{N}_0$, which can be computed from data.

The i.i.d. random process is a common assumption made in control literature, e.g. [28], [24]. It assumes a priori that only the first two moments of the random process are acquired as partial distributional information, which can either be estimated or determined a priori [29].

Assumption 2 (Moment Assumption [30]): There exists a positive α such that $\int_\Xi \exp\left(\|\xi\|^\alpha\right)\mathbb{Q}(\mathrm{d}\xi) < \infty$.

This assumption trivially holds for a bounded uncertainty set Ξ and finite measure $\mathbb{P}$.

Assumption 3 (Polytope Uncertainty Set [17]): The space $\mathscr{M}(\Xi)$ of all probability distributions $\mathbb{Q}$ is supported on a polytope $\Xi := \left\{\xi\in\mathbb{R}^{n_\xi} : C_\xi\xi\leq d_\xi\right\}$.

This assumption means that a shape of the uncertainty set is predefined. This is a common assumption [31] in the context of robust optimization requiring the disturbance not to be infinitely large. This directly stipulates the transformation of distribution into linear inequalities. We subsequently illustrate the equivalent tractable reformulation of the distributionally robust control problem (9).

Theorem 1 (Tractable convex optimization): The optimal control problem (9) with a discounted quadratic cost, distributionally robust constraints within a Wasserstein ball $\mathbb{B}_\varepsilon\left(\widehat{\mathbb{P}}_{N_s}\right)$ centered at the empirical distribution $\widehat{P}_{N_s}$ with N_s samples and radius ε can be reformulated as a cone program (11) using the equivalent affine control laws from lemma 1 and under Assumptions 1-3.

$$\begin{aligned} \inf_{H_N,\lambda,s_i,\gamma_{ik}} \quad & \mathrm{Tr}\left\{\left[(\tilde{C}_x+B_xH_N)^\top J_x(\tilde{C}_x+B_xH_N) + H_N^\top J_u H_N\right]\Sigma_w\right\} \\ & +\mu_w^\top\left[(\tilde{C}_x+B_xH_N)^\top J_x(\tilde{C}_x+B_xH_N) + H_N^\top J_u H_N\right]\mu_w \\ & \text{s.t. } \lambda_j\varepsilon + \tfrac{1}{N}\textstyle\sum_{i=1}^N s_{ij} \leq U_j \\ & b_{tj} + \left\langle a_{tj},\widehat{\xi}_i\right\rangle + \left\langle \gamma_{ijt}, d_\xi - C_\xi\widehat{\xi}_i\right\rangle \leq s_{ij} \\ & \left\|C_\xi^\top\gamma_{ijt} - a_{tj}\right\|_* \leq \lambda_j, \quad \gamma_{ijt}\geq 0 \\ & \forall i\leq N_s, \forall j\leq N_b, \forall t\leq N, \end{aligned} \tag{11}$$

where $J_x := \mathrm{diag}\left(\mathrm{diag}\left(\beta^0,\ldots,\beta^{N-1}\right)\otimes Q, \beta^N Q_f\right)$, $J_u := \mathrm{diag}\left(\beta^0,\ldots,\beta^{N-1}\right)\otimes R$, $a_{tj} = \left[(B_xH_N+\tilde{C}_x)\right]_{tn_x+j,2:Nn_w+1}$ and $b_{tj} = \left[(B_xH_N+\tilde{C}_x)\widehat{\xi}_i\right]_{tn_x+j,1}$. N_s, N_b and N denote sample number, state bound number and length of prediction horizon, respectively. U_j is the bound on state. $\widehat{\xi}_i$ indicates a data point in the training data set, comprising the disturbance sequence consisted of N sampling time.

Proof: We shall prove the equivalence of the objective function and constraints in (10) and (11) respectively. Applying (9) with states stacked over the prediction horizon N from (5) shows that the objective is a minimax expectation of quadratic cost given a disturbance sequence

$$\inf_{H_N}\sup_{\mathbb{P}}\mathbb{E}_{\mathbb{P}\in\mathbb{B}_\varepsilon(\widehat{\mathbb{P}}_{N_s})}\left\{\tilde{w}_{[k,k+N-1]}^\top\left[(\tilde{C}_x+B_xH_N)^\top J_x(\tilde{C}_x+B_xH_N)+H_N^\top J_uH_N\right]\tilde{w}_{[k,k+N-1]}\right\}. \tag{12}$$

Then, under Assumption 1, the mean $\mu_{w_{[k,k+N-1]}}$ and covariance matrix $\Sigma_{w_{[k,k+N-1]}}$ of the i.i.d. disturbance sequence are known/computed from the data, the expectation of the quadratic cost is equivalent to the objective function in (11) according to [32, THEOREM 1.5].

Given constraints in (10), representing the worst-case expectation, the linear combination of states is bounded, and we can therefore prove that they are equivalent to constraints in (11). This is without loss of generality, by proving the equivalence of constraints requiring only

$$\max(\{[x_{[k,k+N]}]_i\}_{i=j}^{j+mn_x})\leq U_j,\forall m\in[0,N], \tag{13}$$

where U_j is a prescribed upper bound on j-th component of state.

Given the stacked state represented by the initial state x_0 and disturbance sequence as in (5), any component of the stacked state within the N-step prediction horizon can be written as

$$[x_{[k,k+N]}]_i=e_i^\top(A^Nx_0+\sum_{n=0}^{N-1}A^nB(h_{N-1-n}+\sum_{\tau=0}^{N-1-n}(H_{n,\tau}(DA^\tau x_0+\sum_{\kappa=0}^{\tau-1}DA^\kappa Cw_{\tau-1-\kappa}+Ew_\tau)))+\sum_{n=0}^{N-1}A^nCw_{N-1-n}). \tag{14}$$

Thus, we then define a pointwise maximum function

$$\ell_j(\xi)=\max(\{[x_{[k,k++N]}]_i\}_{i=j}^{j+mn_x})=\max_{t\leq N}\langle a_{tj},\xi\rangle+b_{tj}, \tag{15}$$

where $a_{tj}=\left[(B_xH_N+\tilde{C}_x)\right]_{tn_x+j,2:Nn_w+1}$ and $b_{tj}=\left[(B_xH_N+\tilde{C}_x)\xi\right]_{tn_x+j,1}$ to shift the maximum value of j-th state entry at each sampling time.

Leveraging the result from [17, Corollary 5.1], the distributionally robust constraints in (10) are rewritten into "best-case" constraints

$$\inf\lambda_j\varepsilon+\frac{1}{N}\sum_{i=1}^{N}s_{ij}\leq U_j \tag{16}$$

along with several additional inequalities. Hence, any feasible solutions of (11) guarantee constraints satisfaction of (8). We thus prove the equivalence of the distributionally robust optimization problem (10) and cone program in the form of (11). ■

Remark 1: The lower bound is constructed by setting $\ell_j(\xi)=\max(\{[-x_{[k,k++N]}]_i\}_{i=j}^{j+mn_x})$ and U_j as negative of the lower bound.

Remark 2: Note that in Section II we require the state constraints to be polyhedral $C_px_{[k,k+N]}\leq D_p$, i.e. only linear combinations of separate state entries. Therefore, ℓ can be selected as an affine function of the states and the distributionally robust constraints and (8), which can therefore be reconstructed by a function ℓ which is affine in the disturbances and initial state. Distributionally robust polyhedral state constraints can then be reformulated into the intersection of linear inequalities as in (11) by considering joint state constraints of several separate state entries effected by a disturbance sequence.

We further prove that the control laws determined by (11) are able to guarantee constraints satisfaction with a finite number of samples.

Theorem 2 (finite sample guarantee [17]): If Assumption 2 (finite moment) holds, and given H_N as the worst-case control law determined via (11) with ambiguity set $\mathbb{B}_{\varepsilon(N_s,\beta)}\left(\widehat{\mathbb{P}}_{N_s}\right)$ and training data set $\widehat{\Xi}_{N_s}$. Then, for any $p\neq n_wN/2$ the finite sample guarantee with confidence level $1-\beta$

$$\mathbb{P}^{N_s}\left\{\mathbb{E}^{\mathbb{P}}[\ell(\xi,H_N)]\leq U\right\}\geq1-\beta \tag{17}$$

holds, where $\beta\in(0,1)$.

Proof: The finite sample guarantee is the simple consequence of [30, Theorem 2]. Under Assumption 2, the probability that the Wasserstein ball radius does not contain the true probability distribution $\mathbb{P}$ is upper bounded by

$$\mathbb{P}^{N_s}\left\{d_{\mathrm{W}}\left(\mathbb{Q},\widehat{\mathbb{P}}_{N_s}\right)\geq\varepsilon\right\}\leq C\exp\left(-cN_s\varepsilon^\kappa\right)\mathbb{I}_{\varepsilon\leq1}+C\exp\left(-cN_s\varepsilon^{\alpha/p}\right)\mathbb{I}_{\varepsilon>1}, \tag{18}$$

where $\kappa(p,\varepsilon)=2$ if $p>n_wN/2$, and $\kappa(p,\varepsilon)=d/p$ if $p\in(0,n_wN/2)$. The positive constants C and c depend only on p,N_s,N,n_w,α.

Let $p=1$ for a type-1 Wasserstein metric, we equate the right-hand side of (18) to β and thus acquire

$$\varepsilon(N_s,\beta)=\begin{cases}\left(\log\left(C\beta^{-1}\right)/(cN_s)\right)^{1/\kappa} & \text{if } N_s\geq\log\left(C\beta^{-1}\right)/c\\ \left(\log\left(C\beta^{-1}\right)/(cN_s)\right)^{1/a} & \text{if } N_s<\log\left(C\beta^{-1}\right)/c.\end{cases} \tag{19}$$

This directly result in

$$\mathbb{P}^{N_s}\left\{\mathbb{P}\in\mathbb{B}_{\varepsilon(N_s,\beta)}\left(\widehat{\mathbb{P}}_{N_s}\right)\geq1-\beta\right\} \tag{20}$$

when N_s is an appropriate finite value. Therefore, $E^{\mathbb{P}}\left[\ell(\xi,H_N)\right]\leq\sup_{\mathbb{Q}\in\mathbb{B}_{\varepsilon(N_s,\beta)}(\widehat{\mathbb{P}}_{N_s})}\mathbb{E}\left[\ell(\xi,H_N)\right]\leq U$ with probability $1-\beta$. ■

Remark 3: The proof of Theorem 2 demonstrates that for any given β, we can guarantee that the true distribution is contained within the Wasserstein ball with confidence level $1-\beta$ if we can either collect sufficient samples or expand the ball radius to be large enough.

Remark 4: For convenience of the proof, we required $p\neq n_wN/2$. However, a similar inequality with $\kappa=\varepsilon/\log(2+1/\varepsilon))^2$ holds for $p=n_wN/2$.

After solving (11) we acquire control laws H_N to govern inputs in the succeeding N steps. By following (3), each input within the prediction horizon can be determined by current and prior purified outputs within the horizon. We can therefore find a policy and then recursively re-solve the conic optimization problem to update our control laws for the subsequent N_u steps. Algorithm 1 illustrates this procedure.

Algorithm 1 Distributionally robust MPC

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Input: A, B, C, D, E, μ_w, Σ_w, J_x, J_u, U_j, ε, C_ξ, d_ξ
Initialize x, C̃_x, B_x, 𝒟, N_s, N, N_b, N_u, a_tj, b_tj, k = 0
repeat {every ΔT}
  if (k mod N_u == 0) then
    Acquire estimated state x_k and update C̃_x, a_tj, b_tj
    Select samples from 𝒟 to formulate Ξ̂_{N_s}
    Solve (11) for H_N
    Denote the sampling time t = k for control laws update
  end if
  Acquire purified observation v_k
  Calculate u_k = h_{k−t} + Σ_{τ=t}^{k} H_{k−t,τ−t} v_τ by using purified observations collected since time instant t of recent control law update
  Store v_k, determine the corresponding w_k and store w_k in 𝒟
  update N_s
  k = k+1
until END
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Remark 5: N_u is the prescribed period of update control laws and it can be selected from $\{1,\ldots,N-1\}$ depending on the severity of disturbance and computational cost. To improve the closed-loop performance, a smaller N_u may be selected, while a larger N_u is chosen to reduce computational cost.

Remark 6: $\mathscr{D}$ is the data set containing individual disturbances collected either prior to or during the process. Each sample $\hat{\xi}_i$ fed to (11) is a disturbance sequence of N individual disturbances, i.e. a window of past instances. It therefore requires that $\mathscr{D}$ contains at least N disturbance data points at initialization. If the disturbance data points are collected in a time sequence, it is possible to replace the oldest data point in the sample $\hat{\xi}_i$ with the newly collected disturbance data point and incorporate one more sample $\hat{\xi}$ into (11) after every N steps. An upper bound may be predefined to limit the maximal number of samples incorporated to determine the control laws to cap the maximal computational cost.

V. SIMULATIONS AND RESULTS

In this section, we test the proposed Algorithm 1 on a disturbed mass spring system. We investigate the impact of data samples on the constraint satisfaction, as well as the finite sample guarantee. We also test this framework on an inverted pendulum system and analyse the influence of various Wasserstein ball radii on state trajectories. We first clarify the simulation details and then compare the performance under a number of settings by analysing their constraint violation.

A. Configuration

We introduce configurations for the subsequent experiments now. Both experiments discretize continuous time models and disturb the systems with sampling period $\Delta T = 0.1\,\text{s}$. The prediction horizon is set to $N = 5$ and each entry of w_k complies with the random process $3\sin(X)$, where $X \sim \mathscr{N}(0,1)$. Therefore, we acquire $C_\xi = \text{diag}(1,\ldots,-1,\ldots)$ and $d_\xi = [3,\ldots,3] \in \mathbb{R}^{2Nn_w}$. The target of both control problems is to steer the disturbed system to the origin whilst satisfying state constraints.

B. Mass spring system

We consider a mass spring system from [33] to illustrate the effectiveness of the proposed Algorithm 1. Maintaining the configuration above, an experiment is conducted with $\varepsilon = 1$ and $N_u = 1$, i.e. the control law is updated at each sampling time. The weighting matrices are given by $Q = \text{diag}(10,1)$, $Q_f = \text{diag}(15,1)$, $R = 1$. Mass position is directly influenced by 1e-3 times disturbance and the measurement of position is also noised by another i.i.d. disturbance scaled by 1e-3, i.e. $C = \begin{bmatrix} \text{1e-3} & 0 \\ 0 & 0 \end{bmatrix}$ and $E = \begin{bmatrix} 0 & \text{1e-3} \end{bmatrix}$. The velocity of mass is upper bounded by $0.4\,\text{m/s}$.

The simulations for Algorithm 1 realize state trajectories with $1,3,5$ collected samples respectively prior to initialization and uses at most 10 recent samples from the data set to avoid large computational costs. As a result, as seen in fig. 1, if we only consider one sample initially, our approach steers the states aggressively. As more samples are used, either collected throughout time or when these samples are provided from the start, the controller steers the state with higher confidence and constraints are satisfied. To analyse our results 50 realizations of the problem are solved. The shaded area of 25-th to 75-th percentiles of simulation initialized with one sample reduces drastically after 1 second - i.e. after collecting two more samples. Additionally, the 75-th percentile of trajectory with $N_{init} = 5$ remains feasible during entire simulation, which also manifest higher confidence of constraints satisfaction after acquiring more samples. This is in line with our theoretical results, the number of samples notably increases confidence on constraints satisfaction, and even with few samples, the approach does not exhibit too much conservativeness. Furthermore, the approach is tractable as a convex cone program is solved.

As demonstrated in fig 2, the ball radius is fixed as 1, the same as in fig 1. We simulate the state trajectory with sample numbers ranging from 1 to 10, each with 50 realizations. The control laws are determined at each sampling time with different samples collected prior to initialization to demonstrate the relation between number of sample and constraint satisfaction. We can read from the figure that the averaged

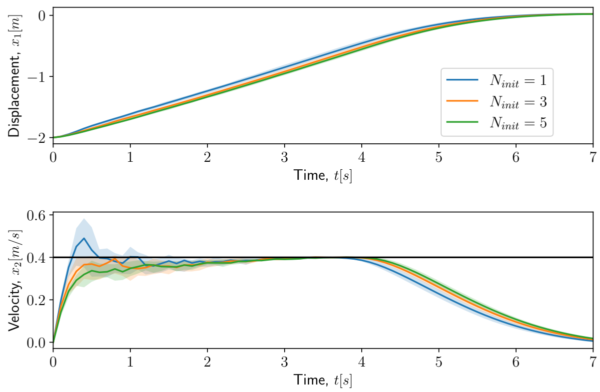

Fig. 1. Simulation results of Algorithm 1 from 50 realizations. The shaded area denotes the 25-th to 75-th trajectory distribution.

trajectory of 50 realizations with only 1 sample tends to violate constraints from the beginning and to oscillate as time increases. In contrast, with large number of samples, constraints are satisfied, this seems to happen for trajectories with sample numbers larger than 5. Furthermore, results from fig 3 illustrate that the confidence of constraints satisfaction is monotonically increasing along the sample number.

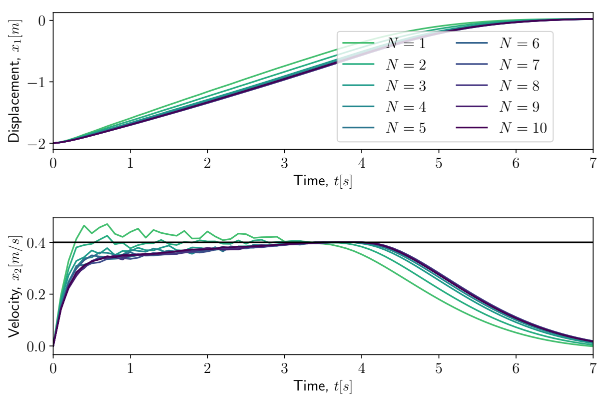

Fig. 2. Simulation results of (11) averaged from 50 realizations with sample number ranging from 1 to 10.

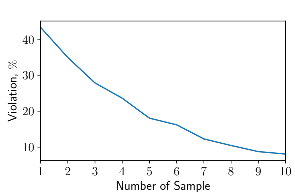

Fig. 3. Relation between sample number and constraint violations within first four seconds, averaged from 50 realizations.

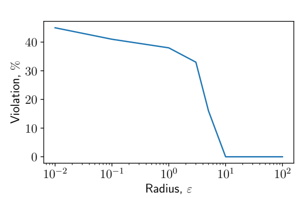

Fig. 4. Relation between Wasserstein ball radius and constraint violations within first two seconds, averaged from 10 realizations.

C. Inverted pendulum

This section illustrates that the constraint violation with a limited number of samples can be mitigated to a significant degree by increasing the Wasserstein ball radius, however, it may leads to conservativeness. We consider an inverted pendulum system represented as the state-space model [34]. Since our interest in this section is to demonstrate the impact of the ball radius, simulations, each with 10 realizations, are carried out for various ball radii, ranging from 0.01 to 100. Control laws are updated at each sampling time with one sample. The weighting matrices Q and Qf are $\text{diag}(1000, 1, 1500, 1)$ and $R = 1$. The pendulum rotational velocity is disturbed by a 1e-2 times disturbance, whereas the measurement of pendulum angular displacement is noised by another i.i.d. disturbance scaled by 1e-2, i.e. $C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \text{1e-2} & 0 \end{bmatrix}$ and $E = \begin{bmatrix} 0 & \text{1e-2} \end{bmatrix}$ The angular velocity is upper bounded by 0.5 1/s.

Same as in the first experiment, we use samples collected prior to the initialization to solve (11) at each sampling time in various settings. As displayed in fig 5, with radius smaller than 1, the state trajectories violates constraints extensively since the center of the ball is roughly located only with one sample and it is very likely that this ball does not contain the true distribution. As the ball radius increases, less constraint violations occur. If a radius of 3 or 5 is used, less constraint violations occur, this guarantees constraints satisfaction with probability 65% and 80% respectively from the simulation. However, if the radius is unnecessarily large, the state trajectories tend to be conservative, e.g. angular velocity varies around 4.7, which is 6% lower than the upper bound. We can also learn from figure 4 that the confidence of probabilistic constraints satisfaction increases as the Wasserstein ball expands when the number of sample is fixed.

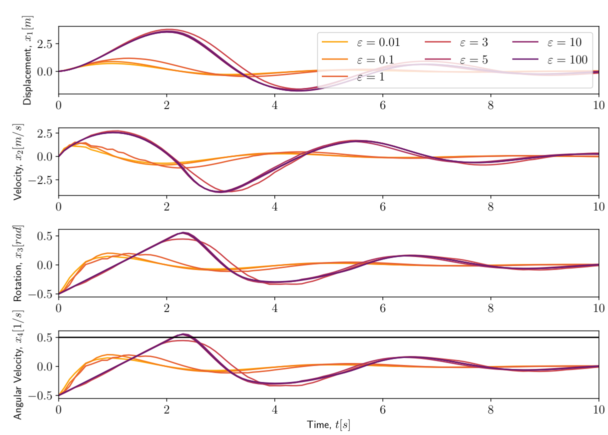

Fig. 5. Simulation results of (11) averaged from 10 realizations with Wasserstein ball radius ranging from 0.01 to 100.

VI. CONCLUSIONS

In this paper, we propose a novel data-driven distributionally robust MPC method for linear systems using the

Wasserstein ball. Our approach relies on building an ambiguity set defined by the Wasserstein metric which allows to characterize the uncertainty even when limited information on the probability distributions is available. In this approach we reformulate the distributionally robust optimal control problem into a tractable convex cone program with finite sample guarantee and propose a practical Algorithm. Numerical case studies on two systems are conducted to illustrate the effectiveness of the Algorithm and to verify the assumptions and theoretical results, such as the finite sample guarantee. Future work includes extending the current approach to incorporate nonlinear dynamics.

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